PROJECTED WRITTEN NOTES FROM THE M 408D LECTURE ON Tuesday, January 23, 2024 on TRIG SUBSTITUTION AND THE LIMIT THEOREM shown in Class

CLASS #3

Sometimes $\sqrt{x^2} \neq x$

If $x = -3$, $x^2 = 9$, $\sqrt{x^2} = \sqrt{9} = 3$

but $x = -3 \neq 3 = \sqrt{x^2}$

Always, $\sqrt{x^2} = |x|$. Here $3 = |-3|$

In this class,

"∞" is a symbol saying

"Some unbounded process is going on."

So, "∞" is not a number.

$\frac{1}{\infty}$ is not a number.

$\infty + \infty$ is not a number.

Review your limit theorems (Sec p. 95).

The Limit Theorem shown in class.

For functions $f(x)$ and $g(x)$,

If $\lim_{x\to a} f(x) = K$, a real number (i.e. finite),

and if $\lim_{x\to a} g(x) = \pm\infty$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

Ex: $\lim_{x\to\infty} \frac{\overset{\to 4}{4}}{\underset{\to\infty}{x}} = 0$ by the Limit Theorem shown in class.

[Do Not write: " $\cancel{\lim_{x\to\infty} \frac{4}{x} = \frac{4}{\infty}} = 0$ ".

Another Ex:

$$\lim_{x\to\infty} \frac{5x+2}{x^2} = \lim_{x\to\infty} \frac{x\left(5+\frac{2}{x}\right)}{x^2}$$

$$= \lim_{x\to\infty} \frac{\overset{\to 5}{\left(5+2/x\right)}}{\underset{\to\infty}{x}} = 0.$$

(by the Limit Theorem shown in class.)

Trig Subsitituteion

Here, we use "$x = \text{Trig func}(\theta)$"
and "$dx = (\text{Trig func}(\theta))' \, d\theta$"

Trig Substitution is appropriate when you have one of these is in your integrand, ("a" is constant)

$\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$.

<u>Example:</u> $\displaystyle\int \frac{1}{(16-x^2)^{3/2}}\,dx = \int \frac{1}{\sqrt{16-x^2}^{\,3}}\,dx$

First we ask: "What Trig func(θ) should we substitute for x?"

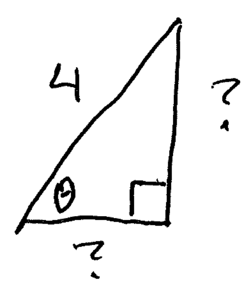
To Answer this (my Way is different from that in the book).

Note that, these problems always have 3 expressions: a, x, Root expression

Here: $a=4$, x, $\sqrt{16-x^2}$ (These are the lengths of sides in a right Δ).

Their Squares: $a^2=16$, x^2, $16-x^2$

which square is the sum of the other two?

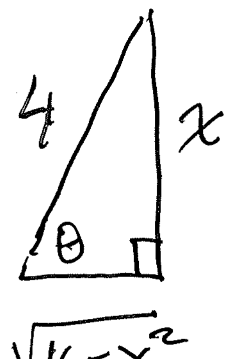


$16 = x^2 + (16-x^2)$

$a=4$ is the length of the hypotenuse

Place a, x, Root expression on the sides so that the Pythagorean Theorem is satisfied.

Rule: When possible, place x on the side opposite θ. Otherwise (ie., when x is on the hypotenuse) place the constant a on the adjacent side to θ.



$$\frac{x}{a} = \frac{x}{4} = \frac{\text{Opp}}{\text{Hyp}} = \sin\theta$$

$$x = 4\sin\theta$$

$$dx = 4\cos\theta\, d\theta$$

$$\frac{\sqrt{16-x^2}}{a} = \frac{\sqrt{16-x^2}}{4} = \frac{\text{Adj}}{\text{Hyp}} = \cos\theta$$

$$\sqrt{16-x^2} = 4\cos\theta$$

$$\int \frac{1}{(\sqrt{16-x^2})^3}\,dx = \int \frac{1}{(4\cos\theta)^3}(4\cos\theta\, d\theta)$$

Don't forget the dx!

$$= \int \frac{4}{64}\left(\frac{\cos\theta}{\cos^3\theta}\right)d\theta$$

$$= \frac{1}{16}\int \frac{1}{\cos^2\theta}\,d\theta$$

To Repeat:

$$\int \frac{1}{(\sqrt{16-x^2})^3}\,dx = \int \frac{1}{(4\cos\theta)^3}\,(4\cos\theta\, d\theta)$$

$\uparrow$ dx

Don't forget the dx!!

$$= \int \frac{4}{64}\left(\frac{\cos\theta}{\cos^3\theta}\right)d\theta$$

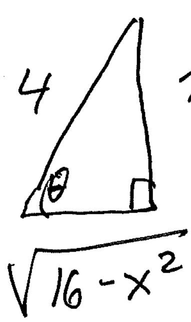
$$= \frac{1}{16}\int \frac{1}{\cos^2\theta}\,d\theta = \frac{1}{16}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{16}\tan\theta + C$$

$$= \frac{1}{16}\left(\frac{x}{\sqrt{16-x^2}}\right) + C$$

$$\int \frac{1}{(\sqrt{16-x^2})^3}\,dx = \frac{x}{16\sqrt{16-x^2}} + C$$

Refer back to the right triangle above!



$$\tan\theta = \frac{opp}{adj} = \frac{x}{\sqrt{16-x^2}}$$

Problem:

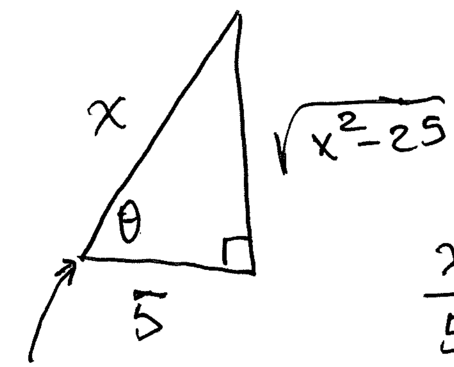
$$\int \frac{\sqrt{x^2-25}}{x}\,dx$$

a, x, Root Exp.

$5,\ x,\ \sqrt{x^2-25}$

Their Squares: $25,\ x^2,\ (x^2-25)$

$$x^2 = (x^2-25)+25$$



$\theta = \sec^{-1}\left(\frac{x}{5}\right)$.

$$\frac{x}{5} = \frac{HYP}{ADJ} = \sec\theta,\quad x = 5\sec\theta.$$

$$dx = 5\sec\theta\tan\theta\,d\theta$$

$$\frac{x}{5} = \sec\theta \Rightarrow \sec^{-1}(\sec\theta) = \theta$$

$$\theta = \sec^{-1}\left(\frac{x}{5}\right)$$

$$\frac{\sqrt{x^2-25}}{5} = \frac{OPP}{ADJ} = \tan\theta$$

$$\sqrt{x^2-25} = 5\tan\theta$$

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{5\tan\theta}{\cancel{5}\sec\theta}(\cancel{5}\sec\theta\tan\theta\,d\theta)$$

$$= 5\int \tan^2\theta\,d\theta = 5\int(\sec^2\theta-1)\,d\theta$$

$$= 5(\tan\theta-\theta)+C \qquad \tan\theta = \frac{OPP}{Adj}$$

$$= 5\left(\left(\frac{\sqrt{x^2-25}}{5}\right) - \sec^{-1}\left(\frac{x}{5}\right)\right)+C = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right)+C$$

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TRIG SUBSTITUTION WITH A DEFINITE INTEGRAL

$$\int_0^1 \frac{dx}{\sqrt{16-x^2}^{\,3}}$$

TO AVOID HAVING TO ADJUST THE LIMITS OF INTEGRATION.

$$\int \frac{1}{\left(\sqrt{16-x^2}\right)^3}\,dx$$

$a=4,\ x,\ \sqrt{16-x^2}$

$x = 4\sin\theta$

$dx = 4\cos\theta$

$\sqrt{16-x^2} = 4\cos\theta$

$$= \frac{x}{16\sqrt{16-x^2}} + C$$

$$\int_0^1 \frac{dx}{\sqrt{16-x^2}^{\,3}}\,dx = \left(\frac{x}{16\sqrt{16-x^2}}\right)\Bigg]_0^1$$

$$= \frac{1}{16\sqrt{15}} - 0 = \underline{\underline{\frac{1}{16\sqrt{15}}}}$$

I Integrating with Partial Fraction Decompositions

(PFDs)

... use for $\int \frac{P(x)}{Q(x)}\,dx$ where $P(x)$ and $Q(x)$ are polynomial.

Ex: $\int \frac{2x-17}{x^2-5x+4}\,dx = \underline{\qquad\qquad}$?

$$\frac{P(x)}{Q(x)} = \frac{2x-17}{x^2-5x+4} \overset{\text{WORK}}{=} \cdots = \frac{5}{x-1} + \frac{-3}{x-4}$$

$$\int \frac{2x-17}{x^2-5x+4}\,dx = \int \left(\frac{5}{x-1} + \frac{-3}{x-4}\right) dx$$

$$= 5\int \frac{1}{x-1}\,dx - 3\int \frac{1}{x-4}\,dx$$

$$= 5\ln(|x-1|) - 3\ln(|x-4|) + C$$

$$= \ln(|x-1|^5) - \ln(|x-4|^3) + C$$

$$= \ln\left(\frac{|x-1|^5}{|x-4|^3}\right) + C$$